Thermodynamics in $f(R,T)$ Theory of Gravity

M. Sharif *and M. Zubair †

Department of Mathematics, University of the Punjab,
Quaid-e-Azam Campus, Lahore-54590, Pakistan.

Abstract

A non-equilibrium picture of thermodynamics is discussed at the apparent horizon of FRW universe in $f(R,T)$ gravity, where R is the Ricci scalar and T is the trace of the energy-momentum tensor. We take two forms of the energy-momentum tensor of dark components and demonstrate that equilibrium description of thermodynamics is not achievable in both cases. We check the validity of the first and second law of thermodynamics in this scenario. It is shown that the Friedmann equations can be expressed in the form of first law of thermodynamics $T_h dS'_h + T_h d_\jmath S' = -dE' + W'dV$, where $d_\jmath S'$ is the entropy production term. Finally, we conclude that the second law of thermodynamics holds both in phantom and non-phantom phases.

Keywords: Modified Gravity; Dark Energy; Apparent Horizon; Thermodynamics.
PACS: 04.50.Kd; 95.36.+x; 97.60.Lf; 04.70.Df.

1 Introduction

Cosmic observations from anisotropy of the Cosmic Microwave Background (CMB) [1], supernova type Ia (SNeIa) [2], large scale structure [3], baryon

*msharif.math@pu.edu.pk
†mzubairkk@gmail.com

acoustic oscillations [4] and weak lensing [5] indicate that expansion of the universe is speeding up rather than decelerating. The present accelerated expansion is driven by gravitationally repulsive dominant energy component known as *dark energy* (DE). There are two representative directions to address the issue of cosmic acceleration. One is to introduce the "exotic energy component" in the context of General Relativity (GR). Several candidates have been proposed [6]-[10] in this perspective to explore the nature of DE. The other direction is to modify the Einstein Lagrangian i.e., modified gravity theory such as $f(R)$ gravity [11].

The discovery of black hole thermodynamics set up a significant connection between gravity and thermodynamics [12]. The Hawking temperature $T=\frac{|\kappa_{sg}|}{2\pi}$, where κ_{sg} is the surface gravity, and horizon entropy $S=\frac{A}{4G}$ obey the first law of thermodynamics [12]-[14]. Jacobson [15] showed that it is indeed possible to derive the Einstein field equations in Rindler spacetime by using the Clausius relation $TdS=\delta Q$ and proportionality of entropy to the horizon area. Here δQ and T are the energy flux across the horizon and Unruh temperature respectively, viewed by an accelerated observer just inside the horizon. Frolov and Kofman [16] employed this approach to quasi-de Sitter geometry of inflationary universe with the relation $-dE=TdS$ to calculate energy flux of slowly rolling background scalar field.

Cai and Kim [17] derived the Freidmann equations of the FRW universe with any spatial curvature from the first law of thermodynamics for the entropy of the apparent horizon. Later, Akbar and Cai [18] showed that the Friedmann equations in GR can be written in the form $dE=TdS+WdV$ at the apparent horizon, where $E=\rho V$ is the total energy inside the apparent horizon and $W=\frac{1}{2}(\rho-p)$ is the work density. The connection between gravity and thermodynamics has been revealed in modified theories of gravity including Gauss-Bonnet gravity [18], Lovelock gravity [19, 20], braneworld gravity [21], non-linear gravity [22]-[27] and scalar-tensor gravity [19, 28, 29]. In $f(R)$ gravity and scalar-tensor theory, non-equilibrium description of thermodynamics is required [22]-[29] so that the Clausius relation is modified in the form $TdS=\delta Q+d\bar{S}$. Here, $d\bar{S}$ is the additional entropy production term.

Recently, Harko et al. [30] generalized $f(R)$ gravity by introducing an arbitrary function of the Ricci scalar R and the trace of the energy-momentum tensor T. The dependence of T may be introduced by exotic imperfect fluids or quantum effects (conformal anomaly). As a result of coupling between

matter and geometry motion of test particles is nongeodesic and an extra acceleration is always present. In $f(R,T)$ gravity, cosmic acceleration may result not only due to geometrical contribution to the total cosmic energy density but it also depends on matter contents. This theory can be applied to explore several issues of current interest and may lead to some major differences. Houndjo [31] developed the cosmological reconstruction of $f(R,T)$ gravity for $f(R,T)=f_1(R)+f_2(T)$ and discussed transition of matter dominated phase to an acceleration phase.

In a recent paper [26], Bamba and Geng investigated laws of thermodynamics in $f(R)$ gravity. It is argued that equilibrium description exists in $f(R)$ gravity. The equilibrium description of thermodynamics in modified gravitational theories is still under debate as various alternative treatments [32] have been proposed to reinterpret the non-equilibrium picture. These recent studies have motivated us to explore whether the equilibrium description can be obtained in the framework of $f(R,T)$ gravity. The study of connection between gravity and thermodynamics in $f(R,T)$ may provide some specific results which would discriminate this theory from various theories of modified gravity.

In this paper, we examine whether an equilibrium description of thermodynamics is possible in such a modified theory of gravity. The horizon entropy is constructed from the first law of thermodynamics corresponding to the Friedmann equations. We explore the generalized second law of thermodynamics (GSLT) and find out the necessary condition for its validity. The paper is organized as follows: In the next section, we review $f(R,T)$ gravity and formulate the field equations of FRW universe. Section **3** investigates the first and second laws of thermodynamics. In section **4**, the Friedmann equations are reformulated by redefining the dark components to explore the possible change in thermodynamics. Finally, section **5** is devoted to the concluding remarks.

2 $f(R,T)$ Gravity

The action of $f(R,T)$ theory of gravity is given by [30]

$$\mathcal{A}=\int dx^4\sqrt{-g}\left[\frac{f(R,T)}{16\pi G}+\mathcal{L}_{(matter)}\right], \tag{1}$$

where $\mathcal{L}_{(matter)}$ determines matter contents of the universe. The energy-momentum tensor of matter is defined as [33]

$$T^{(matter)}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_{(matter)})}{\delta g^{\mu\nu}}. \tag{2}$$

We assume that the matter Lagrangian density depends only on the metric tensor components $g_{\mu\nu}$ so that

$$T^{(matter)}_{\mu\nu} = g_{\mu\nu}\mathcal{L}_{(matter)} - \frac{2\partial\mathcal{L}_{(matter)}}{\partial g^{\mu\nu}}. \tag{3}$$

Variation of the action (1) with respect to the metric tensor yields the field equations of $f(R,T)$ gravity as

$$\begin{aligned} & R_{\mu\nu}f_R(R,T) - \frac{1}{2}g_{\mu\nu}f(R,T) + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f_R(R,T) \\ = \ & 8\pi G T^{(matter)}_{\mu\nu} - f_T(R,T)T^{(matter)}_{\mu\nu} - f_T(R,T)\Theta_{\mu\nu}, \end{aligned} \tag{4}$$

where ∇_μ is the covariant derivative associated with the Levi-Civita connection of the metric and $\Box = \nabla_\mu\nabla^\mu$. We denote $f_R(R,T) = \partial f(R,T)/\partial R$, $f_T(R,T) = \partial f(R,T)/\partial T$ and $\Theta_{\mu\nu} = \frac{g^{\alpha\beta}\delta T_{\alpha\beta}}{\delta g^{\mu\nu}}$. The choice of $f(R,T)\equiv f(R)$ results in the field equations of $f(R)$ gravity.

The energy-momentum tensor of matter is defined as

$$T^{(matter)}_{\mu\nu} = (\rho_m + p_m)u_\mu u_\nu + p_m g_{\mu\nu}, \tag{5}$$

where u_μ is the four velocity of the fluid. If we take $\mathcal{L}_{(matter)} = -p_m$, then $\Theta_{\mu\nu}$ becomes

$$\Theta_{\mu\nu} = -2T^{(matter)}_{\mu\nu} - p_m g_{\mu\nu}. \tag{6}$$

Consequently, the field equations (4) lead to

$$\begin{aligned} & R_{\mu\nu}f_R(R,T) - \frac{1}{2}g_{\mu\nu}f(R,T) + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f_R(R,T) \\ = \ & 8\pi G T^{(matter)}_{\mu\nu} + T^{(matter)}_{\mu\nu}f_T(R,T) + p_m g_{\mu\nu}f_T(R,T). \end{aligned} \tag{7}$$

The field equations in $f(R,T)$ gravity depend on a source term, representing the variation of the energy-momentum tensor of matter with respect to the metric. We consider only the non-relativistic matter (cold *dark* matter and baryons) with $p_m = 0$, therefore the contribution of T comes only from

ordinary matters. Thus, Eq.(7) can be written as an effective Einstein field equation of the form

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G_{eff}T^{(matter)}_{\mu\nu} + T'^{(d)}_{\mu\nu}, \tag{8}$$

where

$$G_{eff} = \frac{1}{f_R(R,T)}\left(G + \frac{f_T(R,T)}{8\pi}\right)$$

is the effective gravitational matter dependent coupling in $f(R,T)$ gravity and

$$T'^{(d)}_{\mu\nu} = \frac{1}{f_R(R,T)}\left[\frac{1}{2}g_{\mu\nu}(f(R,T) - Rf_R(R,T)) + (\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)f_R(R,T)\right] \tag{9}$$

is the energy-momentum tensor of *dark components*. Here, prime means non-equilibrium description of the field equations.

The FRW universe is described by the metric

$$ds^2 = h_{\alpha\beta}dx^\alpha dx^\beta + \tilde{r}^2 d\Omega^2, \tag{10}$$

where $\tilde{r} = a(t)r$ and $x^0 = t,\ x^1 = r$ with the 2-dimensional metric $h_{\alpha\beta} = diag(-1, a^2/(1-kr^2))$. Here $a(t)$ is the scale factor, k is the cosmic curvature and $d\Omega^2$ is the metric of 2-dimensional sphere with unit radius. In FRW background, the gravitational field equations are given by

$$\begin{aligned} 3\left(H^2 + \frac{k}{a^2}\right) &= 8\pi G_{eff}\rho_m + \frac{1}{f_R}\left[\frac{1}{2}(Rf_R - f) - 3H(\dot{R}f_{RR}\right. \qquad (11) \\ &+ \left.\dot{T}f_{RT})\right], \\ -\left(2\dot{H} + 3H^2 + \frac{k}{a^2}\right) &= \frac{1}{f_R}\left[-\frac{1}{2}(Rf_R - f) + 2H(\dot{R}f_{RR} + \dot{T}f_{RT}) + \ddot{R}f_{RR}\right. \\ &+ \left.\dot{R}^2 f_{RRR} + 2\dot{R}\dot{T}f_{RRT} + \ddot{T}f_{RT} + \dot{T}^2 f_{RTT}\right]. \qquad (12) \end{aligned}$$

These can be rewritten as

$$3\left(H^2 + \frac{k}{a^2}\right) = 8\pi G_{eff}(\rho_m + \rho'_d), \tag{13}$$

$$-2\left(\dot{H} - \frac{k}{a^2}\right) = 8\pi G_{eff}(\rho_m + \rho'_d + p'_d), \tag{14}$$

where ρ'_d and p'_d are the energy density and pressure of *dark components*

$$\begin{aligned}\rho'_d &= \frac{1}{8\pi G\mathcal{F}}\left[\frac{1}{2}(Rf_R-f)-3H(\dot{R}f_{RR}+\dot{T}f_{RT})\right], &(15)\\ p'_d &= \frac{1}{8\pi G\mathcal{F}}\left[-\frac{1}{2}(Rf_R-f)+2H(\dot{R}f_{RR}+\dot{T}f_{RT})+\ddot{R}f_{RR}+\dot{R}^2f_{RRR}\right.\\ &+ \left. 2\dot{R}\dot{T}f_{RRT}+\ddot{T}f_{RT}+\dot{T}^2f_{RTT}\right]. &(16)\end{aligned}$$

Here $\mathcal{F}=1+\frac{f_T(R,T)}{8\pi G}$. The equation of state (EoS) parameter of *dark* fluid ω'_d is obtained as $(p'_d=\omega'_d\rho'_d)$

$$\omega'_d=-1+\frac{\ddot{R}f_{RR}+\dot{R}^2f_{RRR}+2\dot{R}\dot{T}f_{RRT}+\ddot{T}f_{RT}+\dot{T}^2f_{RTT}-H(\dot{R}f_{RR}+\dot{T}f_{RT})}{\frac{1}{2}(Rf_R-f)-3H(\dot{R}f_{RR}+\dot{T}f_{RT})}. \quad (17)$$

The semi-conservation equation of ordinary matter is given by

$$\dot{\rho}+3H\rho=q. \quad (18)$$

The energy-momentum tensor of *dark components* may satisfy the similar conservation laws

$$\begin{aligned}\dot{\rho}_d+3H(\rho_d+p_d) &= q_d, &(19)\\ \dot{\rho}_{tot}+3H(\rho_{tot}+p_{tot}) &= q_{tot}, &(20)\end{aligned}$$

where $\rho_{tot}=\rho_m+\rho_d,\ p_{tot}=p_d$ and $q_{tot}=q+q_d$ is the total energy exchange term and q_d is the energy exchange term of dark components. Substituting Eqs.(13) and (14) in the above equation, we obtain

$$q_{tot}=\frac{3}{8\pi G}(H^2+\frac{k}{a^2})\partial_t\left(\frac{f_R}{\mathcal{F}}\right). \quad (21)$$

The relation of energy exchange term in $f(R)$ gravity can be recovered if $\mathcal{F}=1$. In GR, $q_{tot}=0$ for the choice $f(R,T)=R$.

3 Laws of Thermodynamics

In this section, we examine the validity of the first and second law of thermodynamics in $f(R,T)$ gravity for FRW universe.

3.1 First Law of Thermodynamics

Here we investigate the validity of the first law of thermodynamics in $f(R,T)$ gravity at the apparent horizon of FRW universe. The dynamical apparent horizon is determined by the relation $h^{\alpha\beta}\partial_\alpha\tilde{r}\partial_\beta\tilde{r}=0$ which leads to the radius of apparent horizon for FRW universe

$$\tilde{r}_A=\frac{1}{\sqrt{H^2+\frac{k}{a^2}}}. \tag{22}$$

The associated temperature of the apparent horizon is defined through the surface gravity κ_{sg} as

$$T_h=\frac{|\kappa_{sg}|}{2\pi}, \tag{23}$$

where κ_{sg} is given by [17]

$$\begin{aligned}\kappa_{sg} &= \frac{1}{2\sqrt{-h}}\partial_\alpha(\sqrt{-h}h^{\alpha\beta}\partial_\beta\tilde{r}_A)=-\frac{1}{\tilde{r}_A}(1-\frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A})\\ &= -\frac{\tilde{r}_A}{2}(2H^2+\dot{H}+\frac{k}{a^2}).\end{aligned} \tag{24}$$

In GR, the horizon entropy is given by the Bekenstein-Hawking relation $S_h=A/4G$, where $A=4\pi\tilde{r}_A^2$ is the area of the apparent horizon [12]-[14]. In the context of modified gravitational theories, Wald [34] proposed that entropy of black hole solutions with bifurcate Killing horizons is a Noether charge entropy. It depends on the variation of Lagrangian density of modified gravitational theories with respect to Riemann tensor. Wald entropy is equal to quarter of horizon area in units of effective gravitational coupling i.e, $S'_h=A/4G_{eff}$ [35]. In $f(R,T)$ gravity, the Wald entropy is expressed as

$$S'_h=\frac{Af_R}{4G\mathcal{F}}. \tag{25}$$

Taking the time derivative of Eq.(22) and using (14), it follows that

$$f_Rd\tilde{r}_A=4\pi G\tilde{r}_A^3(\rho'_{tot}+p'_{tot})H\mathcal{F}dt, \tag{26}$$

where $\rho'_{tot}=\rho'_m+\rho'_d$ and $p'_{tot}=p'_d$. $d\tilde{r}_A$ is the infinitesimal change in radius of the apparent horizon during a time interval dt. Using Eqs.(25) and (26), we obtain

$$\frac{1}{2\pi\tilde{r}_A}dS'_h=4\pi\tilde{r}_A^3(\rho'_{tot}+p'_{tot})Hdt+\frac{\tilde{r}_A}{2G\mathcal{F}}df_R+\frac{\tilde{r}_Af_R}{2G}d\left(\frac{1}{\mathcal{F}}\right). \tag{27}$$

If we multiply both sides of this equation with a factor $(1-\dot{\tilde{r}}_A/2H\tilde{r}_A)$, it follows that

$$\begin{aligned} T_h dS'_h &= 4\pi\tilde{r}_A^3(\rho'_{tot}+p'_{tot})Hdt - 2\pi\tilde{r}_A^2(\rho'_{tot}+p'_{tot})d\tilde{r}_A + \frac{\pi\tilde{r}_A^2 T_h df_R}{G\mathcal{F}} \\ &+ \frac{\pi\tilde{r}_A^2 T_h f_R}{G} d\left(\frac{1}{\mathcal{F}}\right). \end{aligned} \tag{28}$$

Now, we define energy of the universe within the apparent horizon. The Misner-Sharp energy [36] is defined as $E=\frac{\tilde{r}_A}{2G}$ which can be written in $f(R,T)$ gravity as [24, 28]

$$E' = \frac{\tilde{r}_A}{2G_{eff}}. \tag{29}$$

In terms of volume $V=\frac{4}{3}\pi\tilde{r}_A^3$, we obtain

$$E' = \frac{3V}{8\pi G_{eff}}\left(H^2+\frac{k}{a^2}\right) = V\rho'_{tot} \tag{30}$$

which represents the total energy inside the sphere of radius $\tilde{r}_A$. It is obvious that $E'>0$, if $G_{eff}=\frac{G\mathcal{F}}{f_R}>0$ so that the effective gravitational coupling constant in $f(R,T)$ gravity should be positive. It follows from Eqs.(13) and (30) that

$$dE' = -4\pi\tilde{r}_A^3(\rho'_{tot}+p'_{tot})Hdt + 4\pi\tilde{r}_A^2\rho'_{tot}d\tilde{r}_A + \frac{\tilde{r}_A df_R}{2G\mathcal{F}} + \frac{\tilde{r}_A f_R}{2G} d\left(\frac{1}{\mathcal{F}}\right). \tag{31}$$

Using Eq.(31) in (28), it follows that

$$T_h dS'_h = -dE' + W'dV + \frac{(1+2\pi\tilde{r}_A T_h)\tilde{r}_A df_R}{2G\mathcal{F}} + \frac{(1+2\pi\tilde{r}_A T_h)\tilde{r}_A f_R}{2G} d\left(\frac{1}{\mathcal{F}}\right), \tag{32}$$

which includes the work density $W'=\frac{1}{2}(\rho'_{tot}-p'_{tot})$ [37]. This can be rewritten as

$$T_h dS'_h + T_h d_\jmath S'_h = -dE' + W'dV, \tag{33}$$

where

$$d_\jmath S'_h = -\frac{\tilde{r}_A}{2GT_h}(1+2\pi\tilde{r}_A T_h) d\left(\frac{f_R}{\mathcal{F}}\right) = -\frac{\mathcal{F}(E'+S'_h T_h)}{T_h f_R} d\left(\frac{f_R}{\mathcal{F}}\right). \tag{34}$$

When we compare the cosmological setup of $f(R,T)$ gravity with GR, Gauss-Bonnet gravity and Lovelock gravity [18]-[20], we obtain an auxiliary term

in the first law of thermodynamics. This additional term $d_{\jmath}S'_h$ may be interpreted as entropy production term developed due to the non-equilibrium framework in $f(R,T)$ gravity. This result corresponds to the first law of thermodynamics in non-equilibrium description of $f(R)$ gravity [26] for $f(R,T) = f(R)$. If we assume $f(R,T) = R$, then the traditional first law of thermodynamics in GR can be achieved.

3.2 Generalized Second Law of Thermodynamics

Recently, the GSLT has been studied in the context of modified gravitational theories [25]-[28]. It may be interesting to investigate its validity in $f(R,T)$ gravity. For this purpose, we have to show that [28]

$$\dot{S}'_h + d_{\jmath}\dot{S}'_h + \dot{S}'_{tot} \geq 0, \tag{35}$$

where S'_h is the horizon entropy in $f(R,T)$ gravity, $d_{\jmath}\dot{S}'_h = \partial_t(d_{\jmath}S'_h)$ and S'_{tot} is the entropy due to all the matter and energy sources inside the horizon. The Gibb's equation including all matter and energy fluid is given by [38]

$$T_{tot}dS'_{tot} = d(\rho'_{tot}V) + p'_{tot}dV, \tag{36}$$

where T_{tot} is the temperature of total energy inside the horizon. We assume that T_{tot} is proportional to the temperature of apparent horizon [25, 28], i.e., $T_{tot} = bT_h$, where $0 < b < 1$ to ensure that temperature being positive and smaller than the horizon temperature.

Substituting Eqs.(33) and (36) in Eq.(35), we obtain

$$\dot{S}'_h + d_{\jmath}\dot{S}'_h + \dot{S}'_{tot} = \frac{24\pi\Xi}{\tilde{r}_A bR} \geq 0, \tag{37}$$

where

$$\Xi = (1-b)\dot{\rho}'_{tot}V + (1-\frac{b}{2})(\rho'_{tot} + p'_{tot})\dot{V}$$

is the universal condition to protect the GSLT in modified gravitational theories [28]. Using Eqs.(13) and (14), condition (35) is reduced to

$$\frac{12\pi\mathcal{X}}{bRG\mathcal{F}(H^2 + \frac{k}{a^2})^2} \geq 0, \tag{38}$$

where

$$\begin{aligned}\mathcal{X} &= 2(1-b)H(\dot{H}-\frac{k}{a^2})(H^2+\frac{k}{a^2})f_R+(2-b)H(\dot{H}-\frac{k}{a^2})^2f_R\\&+ (1-b)(H^2+\frac{k}{a^2})^2\mathcal{F}\partial_t(\frac{f_R}{\mathcal{F}}).\end{aligned}$$

Thus the condition to satisfy the GSLT is equivalent to $\mathcal{X}\geq 0$. In flat FRW universe, the GSLT is valid with the constraints $\partial_t(\frac{f_R}{\mathcal{F}})\geq 0$, $H>0$ and $\dot{H}\geq 0$. Also, $\mathcal{F}$ and f_R are positive in order to keep $E>0$. If $b=1$, i.e., temperature between outside and inside the horizon remains the same then the GSLT is valid only if

$$\mathcal{J}=\left(\dot{H}-\frac{k}{a^2}\right)^2\frac{f_R}{\mathcal{F}}\geq 0. \tag{39}$$

For Eq.(10), the effective EoS is defined as $\omega_{eff}=-1-2(\dot{H}-\frac{k}{a^2})/3(H^2+\frac{k}{a^2})$. Here $\dot{H}<\frac{k}{a^2}$ corresponds to quintessence region while $\dot{H}>\frac{k}{a^2}$ represents the phantom phase of the universe. It follows that GSLT in $f(R,T)$ gravity is satisfied in both phantom and non-phantom phases. This result is compatible with [39] according to which entropy may be positive even at the phantom era. Bamba and Geng [26, 27] also shown that second law of thermodynamics can be satisfied in $f(R)$ and $f(T)$ theories of gravity.

4 Redefining the Dark Components

In previous section, we have seen that an additional entropy term $d_\jmath S'_h$ is produced in laws of thermodynamics. This can be considered as the result of non-equilibrium description of the field equations. If we redefine the *dark components* so that the extra entropy production term is vanished, then such formulation is referred as an equilibrium description. It has been seen so far that the equilibrium description does exist in modified theories of gravity [26, 27, 29] and extra entropy production term can be removed.

Here, we discuss whether the equilibrium description of $f(R,T)$ gravity can be anticipated. In fact, we may reduce the entropy production term through this description but it cannot be wiped out entirely. We redefine the energy density and pressure of *dark components*. The (00) and (11)

components of the field equations can be rewritten as

$$3\left(H^2+\frac{k}{a^2}\right) = 8\pi G_{eff}(\rho_m+\rho_d), \tag{40}$$

$$-2\left(\dot{H}-\frac{k}{a^2}\right) = 8\pi G_{eff}(\rho_m+\rho_d+p_d), \tag{41}$$

where $G_{eff}=\left(G+\frac{f_T(R,T)}{8\pi}\right)$ is the effective gravitational coupling, ρ_d and p_d are the energy density and pressure of *dark components* given by

$$\begin{aligned}\rho_d &= \frac{1}{8\pi G\mathcal{F}}\left[\frac{1}{2}(Rf_R-f)-3H(\dot{R}f_{RR}+\dot{T}f_{RT})+3(1-f_R)(H^2\right.\\ &+ \left.\frac{k}{a^2})\right],\end{aligned} \tag{42}$$

$$\begin{aligned}p_d &= \frac{1}{8\pi G\mathcal{F}}\left[-\frac{1}{2}(Rf_R-f)+2H(\dot{R}f_{RR}+\dot{T}f_{RT})+\ddot{R}f_{RR}+\dot{R}^2f_{RRR}\right.\\ &+ \left.2\dot{R}\dot{T}f_{RRT}+\ddot{T}f_{RT}+\dot{T}^2f_{RTT}-(1-f_R)(2\dot{H}+3H^2+\frac{k}{a^2})\right].\end{aligned} \tag{43}$$

The EoS parameter ω_d in this description turns out to be

$$\begin{aligned}\omega_d &= -1+\{\ddot{R}f_{RR}+\dot{R}^2f_{RRR}+2\dot{R}\dot{T}f_{RRT}+\ddot{T}f_{RT}+\dot{T}^2f_{RTT}-H(\dot{R}f_{RR}\\ &+ \dot{T}f_{RT})-2(1-f_R)(\dot{H}-\frac{k}{a^2})\}/\{\frac{1}{2}(Rf_R-f)-3H(\dot{R}f_{RR}+\dot{T}f_{RT})\\ &+ 3(1-f_R)(H^2+\frac{k}{a^2})\},\end{aligned} \tag{44}$$

In this case the expression of total energy exchange is given by

$$q_{tot}=\frac{3}{8\pi G}(H^2+\frac{k}{a^2})\partial_t\left(\frac{1}{\mathcal{F}}\right). \tag{45}$$

Since $\partial_t(f_T(R,T))\neq 0$ in $f(R,T)$ gravity, so that q_{tot} does not vanish. So, we may not establish the equilibrium picture of thermodynamics in this modified gravity. Hence, again we need to consider the non-equilibrium treatment of thermodynamics. This result differ from other modified gravitational theories due to the matter dependence of the Lagrangian density. In $f(R)$ gravity the redefinition of dark components result in local conservation of energy momentum tensor of dark components [26]. It is clear from Eqs.(17) and

(44) that the EoS parameter of *dark components* is not unique in both cases. Thus, one should consider both formulations of the field equations in cosmic discussions.

Now we check the validity of the first and second laws of thermodynamics in this scenario.

4.1 First Law of Thermodynamics

In this representation of the field equations, the time derivative of radius $\tilde{r}_A$ at the apparent horizon is given by

$$d\tilde{r}_A = 4\pi\tilde{r}_A^3 G\mathcal{F}(\rho_{tot} + p_{tot})Hdt. \tag{46}$$

Since in $f(R,T)$ gravity, the equilibrium description is not feasible as it can be seen in modified gravitational theories such that $f(R)$, $f(T)$ and scalar tensor gravity etc. Thus, we use the Wald entropy relation $S_h = A/(4G_{eff})$ rather than introducing Bekenstein-Hawking entropy. Using Eq.(46), the horizon entropy becomes

$$\frac{1}{2\pi\tilde{r}_A}dS_h = 4\pi\tilde{r}_A^3(\rho_{tot} + p_{tot})Hdt + \frac{\tilde{r}_A}{2G}d\left(\frac{1}{\mathcal{F}}\right). \tag{47}$$

The associated temperature of the apparent horizon is

$$T_h = \frac{1}{2\pi\tilde{r}_A}(1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A}). \tag{48}$$

Equations (47) and (48) imply that

$$T_h dS_h = 4\pi\tilde{r}_A^3(\rho_{tot} + p_{tot})Hdt - 2\pi\tilde{r}_A^2(\rho_{tot} + p_{tot})d\tilde{r}_A + \frac{\pi\tilde{r}_A^2 T_h}{G}d\left(\frac{1}{\mathcal{F}}\right). \tag{49}$$

Introducing the Misner-Sharp energy

$$E = \frac{\tilde{r}_A}{2G\mathcal{F}} = V\rho_{tot}, \tag{50}$$

we obtain

$$dE = -4\pi\tilde{r}_A^3(\rho_{tot} + p_{tot})Hdt + 4\pi\tilde{r}_A^2\rho_{tot}d\tilde{r}_A + \frac{\tilde{r}_A}{2G}d\left(\frac{1}{\mathcal{F}}\right). \tag{51}$$

The total work density is defined as [37]

$$W = -\frac{1}{2}T^{(tot)\alpha\beta}h_{\alpha\beta} = \frac{1}{2}(\rho_{tot} - p_{tot}), \tag{52}$$

By combining Eqs.(49), (51) and (52), we obtain the following expression of first law of thermodynamics

$$T_h dS_h + T_h d_\jmath S_h = -dE + WdV, \tag{53}$$

where

$$\begin{aligned} d_\jmath S_h &= -\frac{\tilde{r}_A}{2T_h G}(1 + 2\pi\tilde{r}_A T_h) d\left(\frac{1}{\mathcal{F}}\right) = -\mathcal{F}\left(\frac{E}{T_h} + S_h\right) d\left(\frac{1}{\mathcal{F}}\right) \\ &= -\frac{\pi(4H^2 + \dot{H} + 3k/a^2)}{G(H^2 + k/a^2)(2H^2 + \dot{H} + k/a^2)} d\left(\frac{1}{\mathcal{F}}\right) \end{aligned} \tag{54}$$

is the additional term of entropy produced due to the matter contents of the universe. It involves derivative of $f(R,T)$ with respect to the trace of the energy-momentum tensor. Notice that the first law of thermodynamics $T_h dS_h = -dE + WdV$ holds at the apparent horizon of FRW universe in equilibrium description of modified theories of gravity [26, 27, 29]. However, in $f(R,T)$ gravity, this law does not hold due to the presence of an additional term $d_\jmath S_h$. This term vanishes if we take $f(R,T) = f(R)$ which leads to the equilibrium description of thermodynamics in $f(R)$ gravity.

4.2 Generalized Second Law of Thermodynamics

To establish the GSLT in this formulation of $f(R,T)$ gravity, we consider the Gibbs equation in terms of all matter field and energy contents

$$T_{tot} dS_{tot} = d(\rho_{tot} V) + p_{tot} dV, \tag{55}$$

where T_{tot} denotes the temperature of total energy inside the horizon and S_{tot} is the entropy of all the matter and energy sources inside the horizon. In this case, the GSLT can be expressed as

$$\dot{S}_h + d_\jmath \dot{S}_h + \dot{S}_{tot} \geq 0 \tag{56}$$

which implies that

$$\frac{12\pi\mathcal{Y}}{bRG\mathcal{F}(H^2 + \frac{k}{a^2})^2} \geq 0, \tag{57}$$

where

$$\begin{aligned}\mathcal{Y} &= 2(1-b)H(\dot{H}-\frac{k}{a^2})(H^2+\frac{k}{a^2})+(2-b)H(\dot{H}-\frac{k}{a^2})^2\\ &+ (1-b)(H^2+\frac{k}{a^2})^2\mathcal{F}\partial_t(\frac{1}{\mathcal{F}}).\end{aligned}$$

Thus the GSLT is satisfied only if $\mathcal{Y}\geq 0$. In case of flat FRW universe, the GSLT is met with the conditions $\partial_t(\frac{1}{\mathcal{F}})\geq 0$, $H>0$ and $\dot{H}\geq 0$. In thermal equilibrium $b=1$, the above condition is reduced to the following form

$$\mathfrak{B}=\frac{12\pi H\left(\dot{H}-\frac{k}{a^2}\right)^2}{G\left(H^2+\frac{k}{a^2}\right)^2R}\frac{1}{\mathcal{F}}\geq 0, \tag{58}$$

for $V=\frac{4}{3}\pi\tilde{r}_A^3$ and $R=6(\dot{H}+2H^2+k/a^2)$. $\mathfrak{B}\geq 0$ clearly holds when the Hubble parameter and scalar curvature have same signatures. It can be seen that main difference of results of $f(R,T)$ gravity with $f(R)$ gravity is the term $\mathcal{F}=1+\frac{f_T(R,T)}{8\pi G}$. We remark that in both definitions of *dark components*, the GSLT is valid both in phantom and non-phantom phases of the universe.

5 Concluding Remarks

The fact, $f(R,T)$ gravity is the generalization of $f(R)$ gravity is based on coupling between matter and geometry [30]. This theory can be applied to explore several issues of current interest in cosmology and astrophysics. We have discussed the laws of thermodynamic at the apparent horizon of FRW spacetime in this modified gravity. Akbar and Cai [23] have shown that the Friedmann equations for $f(R)$ gravity can be written into a form of the first law of thermodynamics, $dE=TdS+WdV+Td\bar{S}$, where $d\bar{S}$ is the additional entropy term due to non-equilibrium thermodynamics. Bamba and Geng [26, 27] established the first and second laws of thermodynamics at the apparent horizon of FRW universe with both non-equilibrium and equilibrium descriptions.

We have found that the picture of equilibrium thermodynamics is not feasible in $f(R,T)$ gravity even if we specify the energy density and pressure of *dark components* (see Section **4**). Thus the non-equilibrium treatment

is used to study the laws of thermodynamics in both forms of the energy-momentum tensor of *dark components*. In $f(R,T)$ gravity, q_{tot} does not vanish so there exists some energy exchange with the horizon. The non-equilibrium description can be interpreted as due to some energy flow between inside and outside the apparent horizon. The first law of thermodynamics is obtained at the apparent horizon in FRW background for $f(R,T)$ gravity.

We observe that the additional entropy term is produced as compared to GR, Gauss-Bonnet gravity [18], Lovelock gravity [19, 20] and braneworld gravity [21]. The equilibrium and non-equilibrium description of thermodynamics in $f(R)$ gravity can be obtained if the term $f_T(R,T)$ vanishes *i.e.,* Lagrangian density depends only on geometric part. We have established the GSLT with the assumption that the total temperature inside the horizon T_{tot} is proportional to the temperature of the apparent horizon T_h and evaluated its validity conditions. The GSLT in $f(R)$ gravity follows from condition (38) if $\mathcal{F}=1$. It is concluded that in thermal equilibrium, GSLT is satisfied in both phantom and non-phantom phases.

References

[1] Bennett, C.L. et al.: Astrophys. J. Suppl. **148**(2003)1; Spergel, D.N. et al.: Astrophys. J. Suppl. **148**(2003)175; Spergel, D.N. et al.: Astrophys. J. Suppl. **170**(2007)377.

[2] Perlmutter, S. et al.: Astrophys. J. **483**(1997)565; Perlmutter, S. et al.: Nature **391**(1998)51; Perlmutter, S. et al.: Astrophys. J. **517**(1999)565; Riess, A.G. et al.: Astrophys. J. **607**(2004)665; Riess, A.G. et al.: Astrophys. J. **659**(2007)98.

[3] Hawkins, E. et al.: Mon. Not. Roy. Astron. Soc. **346**(2003)78; Tegmark, M. et al.: Phys. Rev. **D69**(2004)103501; Cole, S. et al.: Mon. Not. Roy. Astron. Soc. **362**(2005)505.

[4] Eisentein, D.J. et al.: Astrophys. J. **633**(2005)560.

[5] Jain, B. and Taylor, A.: Phys. Rev. Lett. **91**(2003)141302.

[6] Sahni, V. and Starobinsky, A.A.: Int. J. Mod. Phys. **D9**(2000)373; Sahni, V.: Lect. Notes Phys. **653**(2004)141; Padmanabhan, T.: Gen. Relativ. Gravit. **40**(2008)529.

[7] Caldwell, R.R.: Phys. Lett. **B545**(2002)23; Nojiri, S. and Odintsov, S.D.: Phys. Lett. **B562**(2003)147; ibid. **B565**(2003)1.

[8] Feng, B., Wamg, X.L. and Zhang, X.M.: Phys. Lett. **B607**(2005)35; Guo, Z.K. et al.: Phys. Lett. **B608**(2005)177.

[9] Kamenshchik, A., Moschella, U. and Pasquier, V.: Phys. Lett. **B511**(2001)265; Bento, M.C., Bertolami, O. and Sen, A.A.: Phys. Rev. **D66**(2002)043507.

[10] Padmanabhan, T.: Phys. Rev. **D66**(2002)021301; Sen, A.: JHEP **48**(2002)204.

[11] Nojiri, S. and Odintsov, S.D.: Int. J. Geom. Methods Mod. Phys. **4**(2007)115; Sotiriou, T. P. and S. Liberati.: Ann. Phys. **322**(2007)935; Sotiriou, T.P and Faraoni, V.: Rev. Mod. Phys. **82**(2010)451; De Felice, A. and Tsujikawa, S.: Living Rev. Rel. **13**(2010)3.

[12] Bardeen, J.M., Carter, B. and Hawking, S.W.: Commun. Math. Phys. **31**(1973)161.

[13] Hawking, S.W.: Commun. Math. Phys. **43**(1975)199.

[14] Bekenstein, J.D.: Phys. Rev. **D7**(1973)2333.

[15] Jacobson, T.: Phys. Rev. Lett. **75**(1995)1260.

[16] Frolov, A.V. and Kofman, L.: JCAP **05**(2003)009.

[17] Cai, R.G. and Kim, S.P.: JHEP **02**(2005)050.

[18] Akbar, M. and Cai, R.G.: Phys. Rev. **D75**(2007)084003.

[19] Cai, R.G. and Cao, L.M.: Phys. Rev. **D75**(2007)064008.

[20] Cai, R.G., Cao, L.M., Hu, Y.P. and Kim, S.P.: Phys. Rev. **D78**(2008)124012.

[21] Sheykhi, A., Wang, B. and Cai, R.G.: Nucl. Phys. **B779**(2007)1; ibid. Phys. Rev. **D76**(2007)023515.

[22] Eling, C., Guedens, R. and Jacobson, T.: Phys. Rev. Lett. **86**(2006)121301.

[23] Akbar, M. and Cai, R.G.: Phys. Lett. **B648**(2007)243.

[24] Gong, Y. and Wang, A.: Phys. Rev. Lett. **99**(2007)211301.

[25] Bamba, K. and Geng, C.Q.: Phys. Lett. **B679**(2009)282.

[26] Bamba, K. and Geng, C.Q.: JCAP **06**(2010)014.

[27] Bamba, K. and Geng, C.Q.: JCAP **11**(2011)008.

[28] Wu, S.-F., Wang, B., Yang, G.-H. and Zhang, P.-M.: Class. Quantum Grav. **25**(2008)235018.

[29] Bamba, K., Geng, C.Q. and Tsujikawa, S.: Phys. Lett. **B668**(2010)101.

[30] Harko, T., Lobo, F.S.N., Nojiri, S. and Odintsov, S.D.: Phys. Rev. **D84**(2011)024020.

[31] Houndjo, M.J.S.: Int. J. Mod. Phys. **D**(to appear, 2012), arXiv:1107.3887.

[32] Gong, Y. and Wang, A.: Phys. Rev. Lett. **99**(2007)211301; Eling, C.: JHEP **11**(2008)048; Elizalde, E. and Silva, P.J.: Phys. Rev. **D78**(2008)061501; Wu, S.-F., Ge, X.-H., Zhang, P.-M. and Yang, G.-H.: Phys. Rev. **D81**(2010)044034.

[33] Landau, L.D. and Lifshitz E.M.: *The Classical Theory of Fileds* (Butterworth-Heinemann, 2002).

[34] Wald, R.M.: Phys. Rev. **D48**(1993)3427.

[35] Brustein, R., Gorbonos, D. and Hadad, M.: Phys. Rev. **D79**(2009)044025.

[36] Misner, C.W. and Sharp, D.H.: Phys. Rev. **136**(1964) B571; Bak, D. and Rey, S.-J.: Class. Quantum Grav. **17**(2000)L83.

[37] Hayward, S.A.: Class. Quantum Grav. **15**(1998)3147; Hayward, S.A., Mukohyama, S. and Ashworth, M.: Phys. Lett. **A256**(1999)347.

[38] Izquierdo, G. and Pavon, D.: Phys. Lett. **B633**(2006)420.

[39] Nojiri, S. and Odintsov, S.D.: Phys. Rev. **D72**(2005)023003.